सर्वं मिदमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Major 1st Semester Examination, 2023-24

MTMDSC101T-MATHEMATICS (MAJOR)

ALGEBRA

Time Allotted: 2 Hours Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any *five* questions from the following: $2\times 5=10$

(a) If a is prime to b, prove that a^2 is prime to b.

(b) If α, β, γ are roots of the cubic equation $x^3+px^2+qx+r=0$, find the value of $\sum\alpha^2$.

(c) Prove that $19^{10}\equiv 1 \pmod{181}$.

(d) Find x, y if $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 1 & x & y & 3\end{pmatrix}$ is an odd permutation.

(e) Find all complex numbers z, such that $\exp(2z+1)=i$.

(f) Examine if the relation ρ on the set of integers $\mathbb{Z}$ is an equivalence relation; where

$$\rho=\{(a,b)\in\mathbb{Z}\times\mathbb{Z} : |a-b|\le 3\}$$

(g) Let $f:A\to B$ and $g:B\to C$ be two mappings. If $g\circ f$ is surjective and g is injective, then prove that f is surjective.

(h) If A be a Hermitian matrix, then show that iA is a skew-Hermitian matrix.

2. (a) Prove that $\sin\left[i\log\frac{a-ib}{a+ib}\right]=\frac{2ab}{a^2+b^2}$. 4

(b) If x, y, z are positive real numbers and $x+y+z=1$, prove that 4

$$8xyz\le(1-x)(1-y)(1-z)\le\frac{8}{27}$$

3. (a) State Descartes' rule of signs. Apply it to find the nature of the roots of the equation $x^4+16x^2+7x-11=0$. 1+2

(b) Solve the biquadratic equation $x^4-4x^3+5x+2=0$ by Ferrari's method. 5

4. (a) Use De-Moivre's theorem to prove $\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$. 4

(b) Find the greatest value of xyz where x, y, z are positive real numbers and $xy+yz+zx=27$. 4

5. (a) Let $S=\{x\in\mathbb{R}: -1<x<1\}$. A map $f:\mathbb{R}\to S$ is defined by $f(x)=\frac{x}{1+|x|}$, $x\in\mathbb{R}$. Show that f is a bijection. Determine f^{-1}. 4

(b) Show that the permutation $f=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 3 & 5 & 1 & 6 & 7 & 8 & 2 & 4\end{pmatrix}$, on the set $S=\{1,2,3,4,5,6,7,8\}$ is an odd permutation. Find the order of f. 2+2

6. (a) By Fermat's theorem, show that $a^{12}-b^{12}$ is divisible by 91 if a and b are both relatively prime to 91. 3

(b) State Euler's function $\phi(n)$, where n is a positive integer. If m and n are positive integers such that m is relatively prime to n, then show that $\phi(mn)=\phi(m)\phi(n)$. 1+4

7. (a) Apply Laplace's method along second and third rows to prove that 5

$$\begin{vmatrix} a & b & c & d\\ -b & a & d & -c\\ -c & -d & a & b\\ -d & c & -b & a\end{vmatrix} = (a^2+b^2+c^2+d^2)^2$$

(b) Reduce the matrix 3

$$A=\begin{pmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ -2 & 2 & 1\end{pmatrix}$$

to row-reduced Echelon form and find its rank.

8. (a) If α is a multiple root of order 3 of the equation $x^4+bx^2+cx+d=0$ $(d\neq 0)$, show that $\alpha=-\frac{8d}{3c}$. 4

(b) Find the general solution of $\sinh z = 2i$. 4

9. (a) Find the values of k, for which the system of equations $kx+y+z=1$, $x+ky+z=1$, $x+y+kz=1$ have (i) a unique solution, (ii) no solution and (iii) more than one solution. 4

(b) Verify Cayley-Hamilton theorem for the matrix $A=\begin{pmatrix}1 & 2 & 1\\ 1 & -1 & 1\\ 2 & 3 & -1\end{pmatrix}$. Express A^{-1} as a Polynomial in A and then compute A^{-1}. 4

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7. (a) Apply Laplace's method along second and third rows to prove that 5

$$\begin{vmatrix}a&b&c&d\\-b&a&d&-c\\-c&-d&a&b\\-d&c&-b&a\end{vmatrix}=(a^2+b^2+c^2+d^2)^2$$

(b) Reduce the matrix 3

$$A=\begin{pmatrix}1&2&-2\\2&1&2\\-2&2&1\end{pmatrix}$$

to row-reduced Echelon form and find its rank.

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